New Applications of the Homotopy Analysis Method

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An analytical technique, namely the homotopy analysis method (HAM), is applied using a computerized symbolic computation to find the approximate and exact solutions of nonlinear evolution equations arising in mathematical physics. The HAM is a strong and easy to use analytic tool for nonlinear problems and does not need small parameters in the equations. The validity and reliability of the method is tested by application on three nonlinear problems, namely the Whitham-Broer-Kaup equations, coupled Korteweg-de Vries equation and coupled Burger's equations. Comparisons are made between the results of the HAM with the exact solutions. The method is straightforward and concise, and it can also be applied to other nonlinear evolution equations in physics.

Key words: Homotopy Analysis Method; Nonlinear Evolution Equations; Approximate and Exact Solutions.

1. Introduction

Since the world around us is inherently nonlinear, nonlinear evolution equations are widely used to describe complex phenomena in various fields of sciences, especially in physics, such as plasma physics, fluid mechanics, optical fibers, solid state physics, nonlinear optics. Exact travelling wave solutions of nonlinear evolution equations are one of the fundamental objects of study in mathematical physics. These exact solutions, when they exist, can help to well understand the mechanism of complicated physical phenomena and dynamical processes modelled by these nonlinear evolution equations. In the past several decades, many significant methods [1 – 9], namely the extended tanh function method, F-expansion method, extended mapping method, Jacobi elliptic function method, variational iteration method, Adomian decomposition method, generalized auxiliary equation method have been applied.

In 1992, Liao [10 – 18] employed the basic ideas of homotopy in topology to propose a general analytical method for nonlinear problems, namely the homotopy analysis method (HAM). Based on homotopy of topology, the validity of the HAM is independent of whether or not small parameters exist in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques [19]. The HAM also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, and avoidance of physically unrealistic assumption. Furthermore, the HAM always provides us with a family of solution expressions in the auxiliary parameter h; the convergences region and rate of each solution might be determined conveniently by the auxiliary parameter h.

The present paper is arranged as follows. In Section 2, we simply provide the mathematical framework of the HAM. In Section 3, in order to illustrate the method, three nonlinear evolution equations are investigated. Also, a comparison is made with the excat solutions obtained by other methods. Finally, conclusions are given in Section 4.

2. Homotopy Analysis Method

To illustrate the methodology of the homotopy analysis method [20 – 23], we consider a differential equation in the form

$$N[z(x,t)] = 0,$$

where N is a nonlinear operator, x and t denote independent variables, $z(x,t)$ is a unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the generalizing homotopy method, Liao [18] constructed the so-called zero-order deformation equations

$$(1-p)L[\phi(x,t;p)-z_0(x,t)] = phN[\phi(x,t;p)], \quad (1)$$

where $p \in (0,1)$ is the embedding parameter, $h \neq 0$ a nonzero auxiliary parameter, L an auxiliary linear operator, $z_0(x,t)$ an initial guess of $z(x,t)$, and $\phi(x,t;p)$ a unknown function. It is important, that one has great freedom to choose auxiliary items in the HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(x,t;0) = z_0(x,t), \quad \phi(x,t;1) = z(x,t).$$

Expanding $\phi(x,t;p)$ in a Taylor series with respect to p it follows

$$\phi(x,t;p) = z_0(x,t) + \sum_{m=1}^{\infty} z_m(x,t)p^m, \quad (2)$$

$$z_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}|_{p=0}. \quad (3)$$

The series (2) converges at $p = 1$ and one has

$$z(x,t) = z_0(x,t) + \sum_{m=1}^{\infty} z_m(x,t), \quad (4)$$

which must be one of the solutions of the original nonlinear equation [18]. As long as $h = -1$, (1) reduces to

$$(1-p)L[\phi(x,t;p)-z_0(x,t)] + pN[\phi(x,t;p)] = 0, \quad (5)$$

which is used in the HAM [18], whereas the solution obtained directly, without using a Taylor series [9]. From (3), the governing equation can be deduced from the zero-order deformation (1). Define the vector

$$Z_n = [z_0(x,t), z_1(x,t), z_2(x,t), \ldots, z_n(x,t)],$$

differentiate (1) m times with respect to the embedding parameter p, and then set $p = 0$ and finally divide by $m!$, one has the so-called mth-order deformation equation

$$L[z_m(x,t) - \chi_m z_{m-1}(x,t)] = hR_m(z_{m-1}), \quad (6)$$

$$R_m(z_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}|_{p=0}. \quad (7)$$

The solution of the mth-order deformation (6) is readily found to be

$$z_m(x,t) = \chi_m z_{m-1}(x,t) + hL^{-1}[R_m(z_{m-1})] \quad (8)$$

with $\chi_m = 0$ for $m < 1$, and $\chi_m = 1$ for $m > 1$.

3. New Applications

To illustrate effectiveness of the HAM three nonlinear evolution equations arising in physics are chosen, namely the coupled Burger's equations, Whitham-Broer-Kaup equations, and coupled Korteweg-de Vries (KdV) equations.

3.1. Example 1: The Coupled Burger's Equations

Let us first consider the coupled Burger's equations [24]

$$\begin{aligned} &u_t - u_{xx} - 2uu_x + uv_x + vu_x = 0, \\ &v_t - v_{xx} - 2vv_x + uv_x + vu_x = 0 \end{aligned} \quad (9)$$

with the initial condition [24]

$$u(x,0) = \sin(x), \quad v(x,0) = \sin(x). \quad (10)$$

To investigate the travelling wave solutions of (9), we choose the linear operator

$$L[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t}$$

with the property $L[c] = 0$, where c is constant. By means of (9), we define a system of nonlinear operators as follows:

$$\begin{aligned} &N_1[\phi_1(x,t;p), \phi_2(x,t;p)] = \\ &\frac{\partial \phi_1(x,t;p)}{\partial t} - \frac{\partial^2 \phi_1(x,t;p)}{\partial x^2} \\ &-2\phi_1(x,t;p)\frac{\partial \phi_1(x,t;p)}{\partial x} + \frac{\partial}{\partial x}[\phi_1(x,t;p)\phi_2(x,t;p)], \end{aligned}$$

$$\begin{aligned} &N_2[\phi_1(x,t;p), \phi_2(x,t;p)] = \\ &\frac{\partial \phi_2(x,t;p)}{\partial t} - \frac{\partial^2 \phi_2(x,t;p)}{\partial x^2} \\ &-2\phi_2(x,t;p)\frac{\partial \phi_2(x,t;p)}{\partial x} + \frac{\partial}{\partial x}[\phi_1(x,t;p)\phi_2(x,t;p)]. \end{aligned}$$

With the aid of the above definition, we construct the zero-order deformation equations

$$\begin{aligned} &(1-p)L[\phi_i(x,t;p) - z_{i,0}(x,t)] \\ &\quad = ph_i N_i[\phi_1(x,t;p),\ \phi_2(x,t;p)], \quad i = 1,2. \end{aligned}$$

Obviously, in case of $p=0$ and $p=1$,

$$\begin{aligned}
&\phi_1(x,t;0) = z_{1,0}(x,t) = u(x,0),\\
&\phi_1(x,t;1) = u(x,t),\\
&\phi_2(x,t;0)(x,t) = z_{2,0}(x,t) = v(x,0),\\
&\phi_2(x,t;1) = v(x,t).
\end{aligned}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi_i(x,t;p)$ varies from the initial guess $z_{i,0}(x,t)$ to the solution $u(x,t)$ and $v(x,t)$, for $i=1,2$, respectively. Expanding $\phi_i(x,t;p)$ in a Taylor series with respect to p for $i=1,2$ admits

$$\phi_i(x,t;p) = z_{i,0}(x,t) + \sum_{m=1}^{\infty} z_{i,m}(x,t)p^m, \tag{11}$$

$$z_{i,m}(x,t) = \frac{1}{m!}\frac{\partial^m \phi_i(x,t;p)}{\partial p^m}|_{p=0}. \tag{12}$$

If the auxiliary linear parameter, the initial conditions, and the auxiliary parameters h_i are chosen, the above series converges at $p=1$, and one has

$$u(x,t) = z_{1,0}(x,t) + \sum_{m=1}^{\infty} z_{1,m}(x,t), \tag{13}$$

$$v(x,t) = z_{2,0}(x,t) + \sum_{m=1}^{\infty} z_{2,m}(x,t), \tag{14}$$

which must be one of the solutions of original nonlinear equations [18]. Defining the vectors

$$Z_{i,n} = [z_{i,0}(x,t), z_{i,1}(x,t), \ldots, z_{i,n}(x,t)],\ i=1,2,$$

we obtain the mth-order deformation equations

$$\begin{aligned}
&L[z_{i,m}(x,t) - \chi_m z_{i,m-1}(x,t)]\\
&\quad = h_i R_{i,m}(z_{1,m-1}, z_{2,m-1}), \quad i=1,2,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&R_{1,m}(z_{1,m-1}, z_{2,m-1}) =\\
&\quad \frac{\partial z_{1,m-1}(x,t)}{\partial t} - \frac{\partial^2 z_{1,m-1}(x,t)}{\partial x^2}\\
&\quad -2\sum_{n=0}^{m-1} z_{1,n}(x,t)\frac{\partial z_{1,m-1-n}(x,t)}{\partial x}\\
&\quad +\frac{\partial}{\partial x}\left[\sum_{n=0}^{m-1} z_{1,n}(x,t) z_{2,m-1-n}(x,t)\right],
\end{aligned}$$

$$\begin{aligned}
&R_{2,m}(z_{1,m-1}, z_{2,m-1}) =\\
&\quad \frac{\partial z_{2,m-1}(x,t)}{\partial t} - \frac{\partial^2 z_{2,m-1}(x,t)}{\partial x^2}\\
&\quad -2\sum_{n=0}^{m-1} z_{2,n}(x,t)\frac{\partial z_{2,m-1-n}(x,t)}{\partial x}\\
&\quad +\frac{\partial}{\partial x}\left[\sum_{n=0}^{m-1} z_{1,n}(x,t) z_{2,m-1-n}(x,t)\right].
\end{aligned}$$

Then the solution of the mth-order deformation equations (15) for $m>1$ admits

$$z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) + h_i L^{-1}[R_{i,m}(z_{1,m-1}, z_{2,m-1})]. \tag{16}$$

Knowing the zeroth initial conditions (10) and the recursive relationship (16), the rest of components are:

$$\begin{aligned}
&z_{1,1}(x,t) = h\sin(x)t,\\
&z_{1,2}(x,t) = \frac{1}{2}h\sin(x)t(2+2h+ht),\\
&z_{1,3}(x,t) = \frac{1}{6}h\sin(x)t\big[6+12h+6ht+6h^2+6h^2t\\
&+h^2t^2 - 2h^2\sin(x)t^2 - 6h\sin(x)t - 6h^2\sin(x)t\\
&+2h^2\cos(x)t^2 + 6h\cos(x)t + 6h^2\cos(x)t\big],\\
&z_{2,1}(x,t) = h\sin(x)t,\\
&z_{2,2}(x,t) = \frac{1}{2}h\sin(x)t(2+2h+ht),\\
&z_{2,3}(x,t) = \frac{1}{6}h\sin(x)t(6+12h+6ht\\
&\qquad\qquad + 6h^2 + 6h^2t + h^2t^2).
\end{aligned}$$

In the same manner the rest of the components can obtained. Using a Taylor series with the initial conditions (10), we obtain the closed form solutions of (9) as follows:

$$\begin{aligned}
&u(x,t) = \sin(x-t)\mathrm{e}^t,\\
&v(x,t) = \sin(x-t)\mathrm{e}^t,
\end{aligned} \tag{17}$$

which are exactly the same as obtained by the variational iteration method [24] with the fixed value $h = -1/101$.

3.2. Example 2: The Whitham-Broer-Kaup Equations

A second instructive model are the Whitham-Broer-Kaup equations [25]

$$
\begin{aligned}
&u_t+uu_x+v_x+\beta u_{xx}=0,\\
&v_t+(vu)_x+\alpha u_{xxx}-\beta v_{xx}=0,
\end{aligned}
\tag{18}
$$

which is a complete integrable model and describes the dispersive long wave in shallow water. In system (18), α,β are real constants that represent different dispersive powers. If $\alpha=0,\beta\neq 0$, system (18) becomes the approximate equations for the long wave equation. If $\alpha=1,\beta=0$, system (18) becomes the variant Boussinesq equation. According to the HAM, we choose the initial approximation [25]

$$
\begin{aligned}
&u(x,0)=2R\sqrt{\alpha^2+\beta^2}\tanh(\xi)-\lambda,\\
&v(x,0)=-\left[2R^2\beta\sqrt{\alpha^2+\beta^2}+2R^2\sqrt{\alpha+\beta^2}\right]\sec^2(\xi),\\
&\xi=x+\lambda t,
\end{aligned}
$$

where λ,R are constants, and the linear operator

$$L[\phi(x,t;p)]=\frac{\partial\phi(x,t;p)}{\partial t}.$$

Furthermore, via (18), we define the nonlinear operators

$$
\begin{aligned}
&N_1\left[\phi_1(x,t;p),\phi_2(x,t;p)\right]=\\
&\frac{\partial\phi_1(x,t;p)}{\partial t}+\beta\frac{\partial^2\phi_1(x,t;p)}{\partial x^2}+\frac{\partial\phi_2(x,t;p)}{\partial x}\\
&+\phi_1(x,t;p)\frac{\partial}{\partial x}\phi_1(x,t;p),\\
&N_2[\phi_1(x,t;p),\phi_2(x,t;p)]=\\
&\frac{\partial\phi_2(x,t;p)}{\partial t}-\beta\frac{\partial^2\phi_2(x,t;p)}{\partial x^2}+\alpha\frac{\partial^3\phi_1(x,t;p)}{\partial x^3}\\
&+\frac{\partial}{\partial x}\left[\phi_1(x,t;p)\phi_2(x,t;p)\right].
\end{aligned}
$$

With the assumption $h=1$, we construct the zero-order deformation equations

$$
\begin{aligned}
&(1-p)L[\phi_i(x,t;p)-z_{i,0}(x,t)]\\
&\quad=ph_iN_i[\phi_1(x,t;p),\phi_2(x,t;p)],\quad i=1,2.
\end{aligned}
$$

As long as $p=0$ and $p=1$,

$$
\begin{aligned}
&\phi_1(x,t;0)=z_{1,0}(x,t)=u(x,0),\quad \phi_1(x,t;1)=u(x,t),\\
&\phi_2(x,t;0)=z_{2,0}(x,t)=v(x,0),\quad \phi_2(x,t;1)=v(x,t).
\end{aligned}
$$

Expanding $\phi_i(x,t;p)$ in a Taylor series with respect to p for $i=1,2$ admits

$$u(x,t)=z_{1,0}(x,t)+\sum_{m=1}^{\infty}z_{1,m}(x,t),\tag{19}$$

$$v(x,t)=z_{2,0}(x,t)+\sum_{m=1}^{\infty}z_{2,m}(x,t).\tag{20}$$

The mth-order deformation equations read

$$
\begin{aligned}
&L[z_{i,m}(x,t)-\chi_m z_{i,m-1}(x,t)]\\
&\quad=h_iR_{i,m}[z_{1,m-1},z_{2,m-1}],\quad i=1,2,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&R_{1,m}(z_{1,m-1},z_{2,m-1})=\\
&\frac{\partial z_{1,m-1}(x,t)}{\partial t}+\beta\frac{\partial^2 z_{1,m-1}(x,t)}{\partial x^2}+\frac{\partial^2 z_{2,m-1}(x,t)}{\partial x}\\
&+\sum_{n=0}^{m-1}z_{1,n}(x,t)\frac{\partial z_{1,m-1-n}(x,t)}{\partial x},\\
&R_{2,m}(z_{1,m-1},z_{2,m-1})=\\
&\frac{\partial z_{2,m-1}(x,t)}{\partial t}-\beta\frac{\partial^2 z_{2,m-1}(x,t)}{\partial x^2}+\alpha\frac{\partial^3 z_{1,m-1-n}(x,t)}{\partial x^3}\\
&+\frac{\partial}{\partial x}\left[\sum_{n=0}^{m-1}z_{1,n}(x,t)z_{2,m-1-n}(x,t)\right].
\end{aligned}
$$

Now, the solution of the mth-order deformation equations (21) for $m>1$ becomes

$$
\begin{aligned}
z_{i,m}(x,t)&=\chi_m z_{i,m-1}(x,t)\\
&\quad+h_iL^{-1}[R_{i,m}(z_{1,m-1},z_{2,m-1})],
\end{aligned}
\tag{22}
$$
$$i=1,2.$$

With the knowledge of the recursive relationship (22) and knowing the zeroth components, the explicit forms of $[z_{1,m},z_{2,m}\ (i=1,2,3,\ m=1,2,3,\ m>1)]$ are written in Appendix A. For simplicity we omit them here.

The numerical behaviour of the approximate solutions using the HAM is shown in Figs. 1a and 2a for different values of x and t.

It should be noted that the HAM solutions are equivalent to the exact solution [25]

$$
\begin{aligned}
&u(x,t)=2R\sqrt{\alpha^2+\beta^2}\tanh(x+\lambda t)-\lambda,\\
&v(x,t)=-[2R^2\beta\sqrt{\alpha^2+\beta^2}+2R^2\sqrt{\alpha+\beta^2}]\\
&\qquad\cdot\sec^2(x+\lambda t).
\end{aligned}
\tag{23}
$$

The behaviour of the exact solutions of $u(x,t)$ and $v(x,t)$ is shown in Figs. 1b and 2b.

(a) (b)

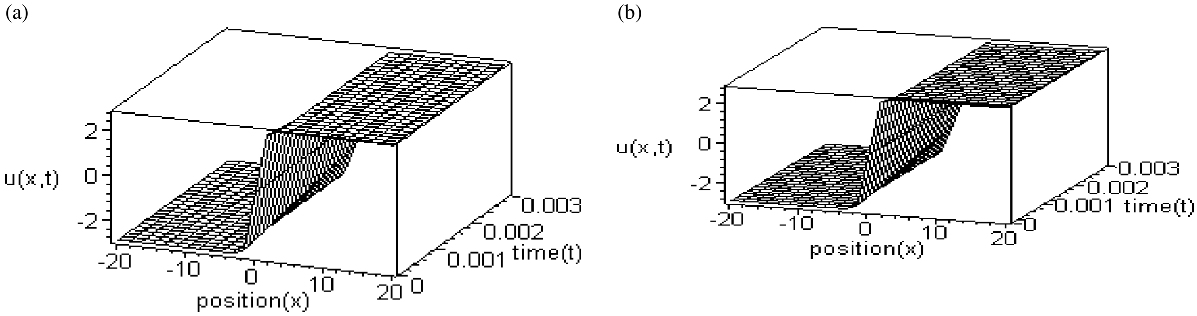

Fig. 1. (a) Approximate solution of $u(x,t)$ of Whitham-Broer-Knaup equations using the HAM with fixed values of $h = -1/101$, $\alpha = 1.0$, $\beta = 1.0$ and $\lambda = 0.1$ for different values of x and t. (b) Exact solution of $u(x,t)$ of Whitham-Broer-Knaup equations with fixed values of $h = -1/101$, $\alpha = 1.0$, $\beta = 1.0$ and $\lambda = 0.1$ for different values of x and t.

(a) (b)

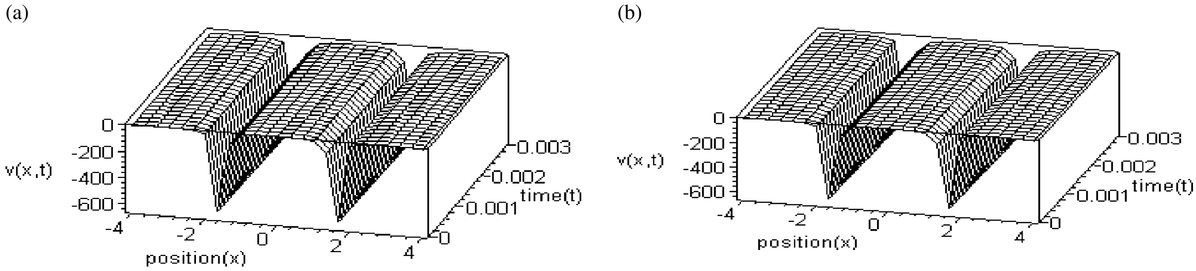

Fig. 2. (a) Approximate solution of $v(x,t)$ of Whitham-Broer-Knaup equations using the HAM with fixed values of $h = -1/101$, $\alpha = 1.0$, $\beta = 1.0$ and $\lambda = 0.1$ for different values of x and t. (b) Exact solution of $v(x,t)$ of Whitham-Broer-Knaup equations with fixed values of $h = -0.01$, $\alpha = 1.0$, $\beta = 1.0$ and $\lambda = 0.1$ for different values of x and t.

3.3. Example 3: The Coupled KdV Equations

In this case, the coupled KdV equations [26] read

$$\begin{aligned} u_t &= a(u_{xxx} + 6uu_x) + 2bvv_x, \\ v_t &= -v_{xxx} - 3uv_x, \end{aligned} \tag{24}$$

where $a \neq -(1/2), ab < 0$, and $M = (-24a/b)^{1/2}k^2$, where k is an arbitrary constant. Proceeding as before to solve (24) by means of the HAM, we choose the initial approximation [26]

$$\begin{aligned} u(x,0) &= -\frac{1+a}{3+6a}k^2 + 4k^2\frac{\mathrm{e}^{kx}}{(1+\mathrm{e}^{kx})^2}, \\ v(x,0) &= \frac{M\mathrm{e}^{kx}}{(1+\mathrm{e}^{kx})^2}, \quad M = \sqrt{-\frac{24a}{b}}k^2. \end{aligned} \tag{25}$$

Following the HAM to solve (24), we define the two operators L and N_i $(i = 1,2)$ as follows:

$$L[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t},$$

$$\begin{aligned} N_1[\phi_1(x,t;p), \phi_2(x,t;p)] &= \frac{\partial \phi_1(x,t;p)}{\partial t} \\ &- a\frac{\partial^3 \phi_1(x,t;p)}{\partial x^3} - 6a\phi_1(x,t;p)\frac{\partial \phi_1(x,t;p)}{\partial x} \\ &-2b\phi_2(x,t;p)\frac{\partial \phi_2(x,t;p)}{\partial x}, \end{aligned}$$

$$N_2[\phi_1(x,t;p), \phi_2(x,t;p)] =$$

$$\frac{\partial \phi_2(x,t;p)}{\partial t} + \frac{\partial^3 \phi_2(x,t;p)}{\partial x^3} + 3\phi_1(x,t;p)\frac{\partial \phi_2(x,t;p)}{\partial x}.$$

The mth-order deformation equations yield

$$\begin{aligned} &L[z_{i,m}(x,t) - \chi_m z_{i,m-1}(x,t)] \\ &\quad = h_i R_{i,m}(z_{1,m-1}, z_{2,m-1}), \quad i = 1,2 \end{aligned} \tag{26}$$

$$\begin{aligned} &R_{1,m}(z_{1,m-1}, z_{2,m-1}) = \\ &\quad \frac{\partial z_{1,m-1}(x,t)}{\partial t} - a\frac{\partial^3 z_{1,m-1}(x,t)}{\partial x^3} \\ &\quad - 6a\sum_{n=0}^{m-1} z_{1,n}(x,t)\frac{\partial z_{1,m-1-n}(x,t)}{\partial x} \\ &\quad - 2b\sum_{n=0}^{m-1} z_{2,n}(x,t)\frac{\partial z_{2,m-1-n}(x,t)}{\partial x}, \end{aligned}$$

(a) (b)

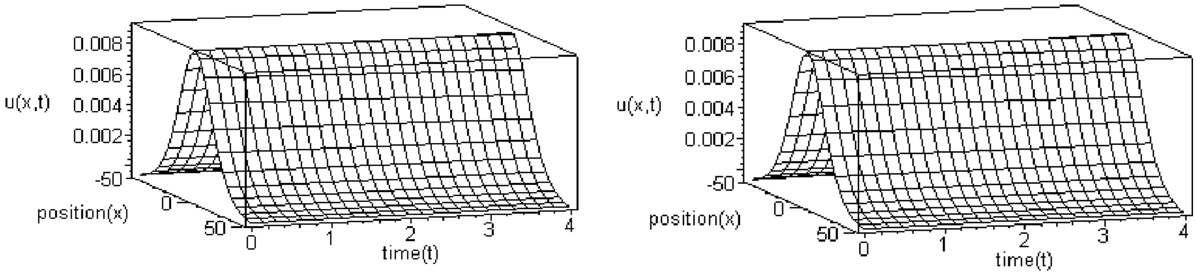

Fig. 3. (a) Approximate solution of $u(x,t)$ of coupled KdV equations using the HAM with fixed values of $h=-0.01$, $a=-1.5$, $c=0.1$, $b=0.1$ and $\kappa=0.1$ for different values of x and t. (b) Exact solution of $u(x,t)$ of coupled KdV equations with fixed values of $h=-0.01$, $a=-1.5$, $c=0.1$, $b=0.1$ and $\kappa=0.1$ for different values of x and t.

(a) (b)

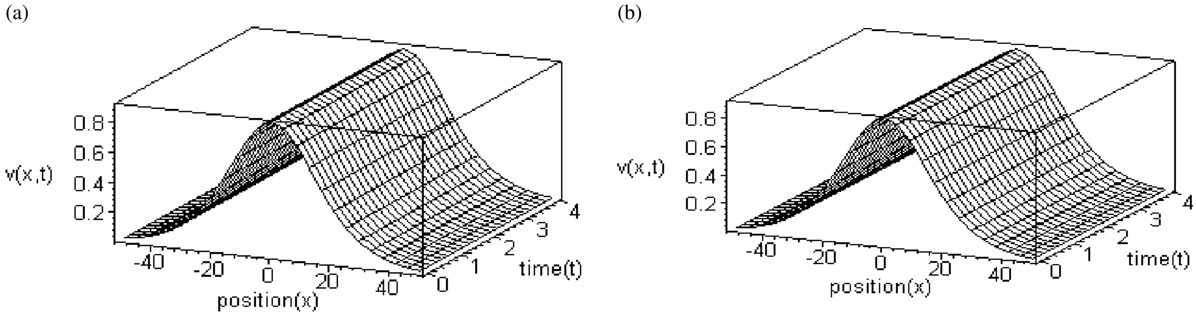

Fig. 4. (a) Approximate solution of $v(x,t)$ of coupled KdV equations using the HAM with fixed values of $h=-0.01$, $a=-1.5$, $c=0.1$, $b=0.1$ and $\kappa=0.1$ for different values of x and t. (b) Exact solution of $v(x,t)$ of coupled KdV equations with fixed values of $h=-0.01$, $a=-1.5$, $c=0.1$, $b=0.1$ and $\kappa=0.1$ for different values of x and t.

$$
\begin{aligned}
&R_{2,m}(z_{1,m-1},z_{2,m-1}) = \\
&\frac{\partial z_{2,m-1}(x,t)}{\partial t} + \frac{\partial^3 z_{2,m-1}(x,t)}{\partial x^3} \\
&+3\sum_{n=0}^{m-1} z_{1,n}(x,t)\frac{\partial z_{2,m-1-n}(x,t)}{\partial x}.
\end{aligned}
$$

The solution of the mth-order deformation becomes

$$
\begin{aligned}
&z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) \\
&+ h_i L^{-1}[R_{i,m}(z_{1,m-1},z_{2,m-1})], \quad i=1,2.
\end{aligned} \tag{27}
$$

By means of the recursive relationships (27) and (25), the rest of the components of $[z_{1,m}, z_{2,m}$ $(i=1,2,3,$ $m=1,2,3,$ $m>1)]$ can be directly evoluted (see Appendix B). The behaviour of the approximate solutions of $u(x,t)$ and $v(x,t)$ with the fixed values $a=-1.5$, $c=0.1$, $b=0.1$, $k=0.1$ and $h=-1/101$ for different values of x and t are shown in Figs. 3a and 4a. Also it is shown from Figs. 3b and 4b that the HAM solutions agree with the exact solution [26]

$$
u(x,t) = -\frac{1+a}{3+6a}k^2 + 4k^2\frac{\mathrm{e}^{k(x+ct)}}{(1+\mathrm{e}^{k(x+ct)})^2},
$$

$$
v(x,t) = \frac{M\mathrm{e}^{k(x+ct)}}{(1+\mathrm{e}^{k(x+ct)})^2}. \tag{28}
$$

4. Conclusion

In this paper, the homotopy analysis method (HAM) is applied for constructing the approximate and exact solutions of three nonlinear evolution equations arising in mathematical physics, namely the coupled Burger's equations, Whitham-Broer-Kaup equations, and coupled KdV equations.

The advantages of the HAM with respect to the homotopy perturbation method are illustrated. The HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between the HAM and other methods.

It is worthwhile to mention that the HAM is straightforward and concise, and it can also be applied to other nonlinear problems in science and engineering. This is our task in the future.

Appendix A

$$z_{1,0} := 2\sqrt{\alpha+\beta^2}\tanh(x)-\lambda, \quad z_{2,0} := -\left(2\beta\sqrt{\alpha+\beta^2}+2\sqrt{\alpha+\beta^2}\right)\sec(x)2,$$

$$\begin{aligned}
z_{1,1} := {} & 2ht\big[-2\sinh(x)\cos(x)^3\beta\sqrt{\alpha+\beta^2}+2\sinh(x)\cos(x)^3\alpha+2\sinh(x)\cos(x)^3\beta^2\\
& -2\sqrt{\alpha+\beta^2}\sin(x)\beta\cosh(x)^3-2\sqrt{\alpha+\beta^2}\sin(x)\beta\cosh(x)^3-\sqrt{\alpha+\beta^2}\lambda\cosh(x)\cos(x)^3\big]\\
& /\left(\cosh(x)^3\cos(x)^3\right),
\end{aligned}$$

$$\begin{aligned}
z_{2,1} := {} & -4ht\big[2\cosh(x)^3\cos(x)\sin(x)\sinh(x)\beta\alpha+2\cosh(x)^3\cos(x)\sin(x)\sinh(x)\beta^3\\
& +2\cosh(x)^3\cos(x)\sin(x)\sinh(x)\alpha+2\cosh(x)^3\cos(x)\sin(x)\sinh(x)\beta^2\\
& -\sqrt{\alpha+\beta^2}\cos(x)\sin(x)\lambda\beta\cosh(x)^4-\sqrt{\alpha+\beta^2}\cos(x)\sin(x)\lambda\cosh(x)^4\\
& -3\sqrt{\alpha+\beta^2}\beta\cosh(x)^4+2\sqrt{\alpha+\beta^2}\cos(x)^2\beta^2\cosh(x)^4+2\sqrt{\alpha+\beta^2}\cos(x)^2\beta\cosh(x)^4\\
& -3\sqrt{\alpha+\beta^2}\beta^2\cosh(x)^4+\cos(x)^2\cosh(x)^2\beta^2-2\sqrt{\alpha+\beta^2}\alpha\cos(x)^4\cosh(x)^2\\
& +\cos(x)^2\cosh(x)^2\beta\alpha+\cos(x)^2\cosh(x)^2\beta^3+\cos(x)^2\cosh(x)^2\alpha+3\sqrt{\alpha+\beta^2}\alpha\cos(x)^4\big]\\
& /\left(\cosh(x)^4\cos(x)^4\right),
\end{aligned}$$

$$\begin{aligned}
z_{1,2} := {} & 2ht\big[-6ht\cosh(x)\cos(x)^4\lambda\alpha+6ht\cosh(x)^5\sqrt{\alpha+\beta^2}\lambda\beta-h\cosh(x)^3\cos(x)^4\sqrt{\alpha+\beta^2}\lambda\\
& +2\cosh(x)^2\cos(x)^4\sinh(x)\beta^2+2\cosh(x)^2\cos(x)^4\sinh(x)\alpha-2\cosh(x)^5\cos(x)\sqrt{\alpha+\beta^2}\sin(x)\\
& -\cosh(x)^3\cos(x)^4\sqrt{\alpha+\beta^2}\lambda+8ht\cosh(x)^4\cos(x)^2\sinh(x)\beta^3-2h\cosh(x)^5\cos(x)\sqrt{\alpha+\beta^2}\sin(x)\\
& -2h\cosh(x)^2\cos(x)^4\sinh(x)\beta\sqrt{\alpha+\beta^2}+8h\beta^3t\cosh(x)^2\cos(x)^4\sinh(x)\\
& +8h\beta t\cosh(x)^2\cos(x)^4\sinh(x)\alpha-8h\beta^2t\cosh(x)^2\cos(x)^4\sinh(x)\sqrt{\alpha+\beta^2}\\
& -20h\beta t\sinh(x)\cos(x)^4\alpha+20h\beta^2t\sinh(x)\cos(x)^4\sqrt{\alpha+\beta^2}-4h\beta t\cosh(x)^3\cos(x)^4\sqrt{\alpha+\beta^2}\lambda\\
& +6h\beta t\cosh(x)\sqrt{\alpha+\beta^2}\lambda\cos(x)^4-12ht\cosh(x)^4\sinh(x)\beta^3-ht\cosh(x)^2\cos(x)^4\sqrt{\alpha+\beta^2}\lambda^2\sinh(x)\\
& +4ht\cosh(x)^3\cos(x)^4\lambda\alpha-6ht\cosh(x)\cos(x)^4\lambda\beta^2-8ht\cosh(x)^2\cos(x)^4\sqrt{\alpha+\beta^2}\alpha\sinh(x)\\
& -4ht\cosh(x)^5\cos(x)^2\sqrt{\alpha+\beta^2}\lambda\beta+4ht\cosh(x)^3\cos(x)^4\lambda\beta^2-2h\cosh(x)^5\cos(x)\sqrt{\alpha+\beta^2}\beta\sin(x)\\
& -2\cosh(x)^2\sinh(x)\beta\sqrt{\alpha+\beta^2}-2\cosh(x)^5\cos(x)\sqrt{\alpha+\beta^2}\beta\sin(x)+2ht\cosh(x)^2\cos(x)^2\beta^3\sinh(x)\\
& -12ht\cosh(x)^4\sinh(x)\alpha+2ht\cosh(x)^2\cos(x)^2\beta\alpha\sinh(x)+6ht\cosh(x)^5\sqrt{\alpha+\beta^2}\lambda\\
& -6ht\cosh(x)^3\cos(x)\alpha\sin(x)+2ht\cosh(x)^2\cos(x)^2\beta^2\sinh(x)-6ht\cosh(x)^3\cos(x)\beta^2\sin(x)\\
& -6ht\cosh(x)^3\cos(x)\beta\alpha\sin(x)+8ht\cosh(x)^4\cos(x)^2\sinh(x)\beta\alpha-12ht\cosh(x)^4\sinh(x)\beta\alpha\\
& -6ht\cosh(x)^3\cos(x)\beta^3\sin(x)+2ht\cosh(x)^2\cos(x)^2\alpha\sinh(x)+8ht\cosh(x)^4\cos(x)^2\sinh(x)\alpha\\
& +8ht\cosh(x)^4\cos(x)^2\sinh(x)\beta^2-4ht\cosh(x)^5\cos(x)^2\sqrt{\alpha+\beta^2}\lambda+20ht\sinh(x)\cos(x)^4\sqrt{\alpha+\beta^2}\alpha\\
& -12ht\cosh(x)^4\sinh(x)\beta^2-20h\beta^3t\sinh(x)\cos(x)^4+2h\cosh(x)^2\cos(x)^4\sinh(x)\beta^2\\
& +2h\cosh(x)^2\cos(x)^4\sinh(x)\alpha\big]/\left(\cos(x)^4\cosh(x)^5\right).
\end{aligned}$$

Appendix B

$$z_{1,0} := -\frac{(1+a)k^2}{3+6a} + \frac{4k^2\,\mathrm{e}^{(kx)}}{\left(1+\mathrm{e}^{(kx)}\right)^2}, \quad z_{2,0} := \frac{M\,\mathrm{e}^{(kx)}}{\left(1+\mathrm{e}^{(kx)}\right)^2},$$

$$z_{1,1} := -2hk\,\mathrm{e}^{(kx)}\big[-2ak^4 + 22ak^4\,\mathrm{e}^{(kx)} + 48a^2k^4\,\mathrm{e}^{(kx)} - 22ak^4\,\mathrm{e}^{(2kx)} - 48a^2k^4\,\mathrm{e}^{(2kx)}$$
$$+2ak^4\,\mathrm{e}^{(3kx)} + bM^2\,\mathrm{e}^{(kx)} + 2bM^2\,\mathrm{e}^{(kx)}a - bM^2\,\mathrm{e}^{(2kx)} - 2bM^2\,\mathrm{e}^{(2kx)}a\big]t\big/\big[(1+2a)(1+\mathrm{e}^{(kx)})^5\big],$$

$$z_{2,1} := -\frac{t(-1+\mathrm{e}^{(kx)})a\,\mathrm{e}^{(kx)}k^3Mh}{(1+\mathrm{e}^{(kx)})^3(1+2a)},$$

$$z_{2,2} := -\frac{1}{2}t\,\mathrm{e}^{(kx)}k^2Mh\big[-48h\,\mathrm{e}^{(3kx)}tbM^2a - htk^4a^2 - 2hak - 4ha^2k + 576h\,\mathrm{e}^{(4kx)}ta^3k^4 + 2ak\,\mathrm{e}^{(6kx)}$$
$$+4a^2k\,\mathrm{e}^{(6kx)} + 16a^2k\,\mathrm{e}^{(5kx)} - 1152h\,\mathrm{e}^{(3kx)}ta^3k^4 + 24h\,\mathrm{e}^{(4kx)}tbM^2a + 4ha^2k\,\mathrm{e}^{(6kx)} - 8ak\,\mathrm{e}^{(kx)} - 16a^2k\,\mathrm{e}^{(kx)}$$
$$-48h\,\mathrm{e}^{(3kx)}tbM^2a^2 - 8hak\,\mathrm{e}^{(kx)} - 16ha^2k\,\mathrm{e}^{(kx)} - 10ak\,\mathrm{e}^{(2kx)} - 20a^2k\,\mathrm{e}^{(2kx)}tbM^2 + 6h\,\mathrm{e}^{(4kx)}tbM^2$$
$$+10ak\,\mathrm{e}^{(4kx)} + 20a^2k\,\mathrm{e}^{(4kx)} + 24h\,\mathrm{e}^{(2kx)}tbM^2a^2 + 144htk^4a\,\mathrm{e}^{(2kx)} + 20ha^2k\,\mathrm{e}^{(4kx)} + 8hak\,\mathrm{e}^{(5kx)}$$
$$+16ha^2k\,\mathrm{e}^{(5kx)} - 10hak\,\mathrm{e}^{(2kx)} + 8ak\,\mathrm{e}^{(5kx)} + 585htk^4a^2\,\mathrm{e}^{(2kx)} - 20ha^2k\,\mathrm{e}^{(2kx)} + 10hak\,\mathrm{e}^{(4kx)}$$
$$+6h\,\mathrm{e}^{(2kx)}tbM^2 - 2ak - 4a^2k - 288htk^4a\,\mathrm{e}^{(3kx)} - 1136htk^4a^2\,\mathrm{e}^{(3kx)} - 12h\,\mathrm{e}^{(3kx)}tbM^2$$
$$+144htk^4a\,\mathrm{e}^{(4kx)} + 585htk^4a^2\,\mathrm{e}^{(4kx)} - htk^4a^2\,\mathrm{e}^{(6kx)} + 24h\,\mathrm{e}^{(4kx)}tbM^2a^2 + 576h\,\mathrm{e}^{(2kx)}ta^3k^4$$
$$+24h\,\mathrm{e}^{(2kx)}tbM^2a + 2hak\,\mathrm{e}^{(6kx)}\big]\big/\big[(1+\mathrm{e}^{(kx)})^8(1+2a)^2\big].$$

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